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A NOTE ON THE RAYLEIGH HYPOTHESIS AND THE AKI-LARNER METHOD

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The seismic response of laterally irregular stratified media has been studied by many authors (for recent reviews, see, *e.g.*, Sánchez-Sesma, 1987; Aki, 1988). Significant advances have been obtained since the pioneering work of Aki and Larner (1970). In their method, the diffracted and refracted fields are represented by superposition of plane waves, including inhomogeneous waves, propagating in many directions. The total motion is obtained through integration over horizontal wavenumber. Under the assumption of horizontal periodicity of the structure, the integral is replaced by an infinite sum. Truncation of this sum and spatial Fourier transformation of boundary conditions lead to a system of linear equations for the complex coefficients of the horizontal wavenumber expansion. The Aki-Larner discrete wavenumber method has found many applications in seismology, due mainly to its flexibility to model elastic wave fields (see e.g., Bouchon, 1973; Bouchon and Aki, 1977a, b; Bouchon, 1979; Bard and Bouchon, 1980a, b; Bard, 1982; Bard and Gariel, 1986). Its major disadvantage, namely the difficulty to model wave fields near very steep interfaces, has been corrected by Bouchon (1985) and Campillo and Bouchon (1985), who used a *single-layer* expansion of the fields, similar to the one used by Sánchez-Sesma and Esquivel (1979). Bouchon and Campillo constructed the full-space Green's function by discrete horizontal wavenumber summation. In their treatment, the sources are located along the interface and the truncation of the sums guarantees a regular field representation everywhere. On the other hand, Kawase (1988) used a discrete wavenumber representation for the Green's function and a rigorous boundary element method formulation. He carried out analytical integrations in the boundary elements and his results for the surface response of a semicircular canyon on a half-space are in excellent agreement with the analytical solution for *SH* waves (Trifunac, 1973) and with numerical results for *P*, *SV*, and Rayleigh waves (Wong, 1982; Sánchez-Sesma *et al.*, 1985; Dravinski and Mossessian, 1987).

The failure of the original Aki and Larner (1970) method to accurately represent wave fields close to large-slope interfaces has been attributed to the limitation imposed by the so-called "Rayleigh hypotesis," which consists in representing the diffracted fields, say, in an irregular surface half-space with an integral in the horizontal wavenumber which includes downgoing waves and inhomogeneous plane waves. That integral does not include *explicitly* upgoing waves, therefore this fact has been traditionally considered the cause of the failure (see, e.g., Aki and Richards, 1980).

In this note, we show that the Rayleigh ansatz is quite good and that the reasons for the failure lie elsewhere. They are of numerical nature. Our purpose in this note is to contribute to a better understanding of this powerful technique. In what follows, we consider the problem of incident plane *SH* waves upon a semicircular canyon on the surface of a half-space and write the Trifunac (1973) exact solution as an integral in the horizontal wavenumber. The diffracted part of the solution is represented with explicit *downgoing* waves plus inhomogeneous waves. This makes clear that upgoing diffracted energy admits a representation in terms of inhomogeneous waves only.

Exact Solution

Consider the semicircular canyon of radius a on the surface of a homogeneous, isotropic, elastic half-space shown in Figure 1 and assume incidence of harmonic plane *SH* waves with incidence angle θ_0. In this case, the equation of motion for the antiplane displacement v is given by

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} + q^2 v = 0 \tag{1}$$

where $q = \omega/\beta$, ω = circular frequency and β = shear-wave velocity. Free-boundary conditions imply zero normal derivative at the irregular surface. This and the Sommerfeld (1949) radiation condition for the diffracted field are fulfilled by the exact solution (Trifunac, 1973) which is given by

$$v = v^i + v^r - 2 \sum_{m=0}^{\infty} b_m H_m^{(1)}(qr)\cos m\theta \tag{2}$$

where $v^{i,r} = \exp(ik_0 x \mp i\gamma_0 z)$ are the incident and reflected waves for the free-field solution (it is assumed here and hereafter that the time dependence is of the form $e^{-i\omega t}$), $k_0 = q\cos\theta_0$, $\gamma_0 = q\sin\theta_0$, $r = (x^2 + z^2)^{1/2}$, $\tan\theta = z/x$, $b_m = \epsilon_m i^m \cos m\theta_0 J'_m(qa)/H_m^{(1)\prime}(qa)$, ϵ_m = Neumann factor (=1 if $m = 0$; = 2 if $m > 0$), $J_m(\cdot)$ = Bessel function of the first kind and order m and $H_m^{(1)}(\cdot)$ = Hankel function of the first kind and order m. The primes mean derivative with respect to the argument. This exact solution has been computed by Trifunac (1973) up to a normalized frequency of $qa = 3\pi$. An asymptotic evaluation for higher frequencies (for a related problem of electromagnetic waves) is due to Franz (1954) who discovered the so-called *creeping* waves. These waves are included in the diffracted part of the field and have been recently computed by Kawase (1988).

Some Horizontal Wavenumber Integrals

It is well known that the Hankel function of zero order can be written by means of (Lamb, 1904)

$$H_0^{(1)}(qr) = \frac{1}{\pi} \int_{-\infty}^{\infty} \exp(ikx + i\gamma z)\, \frac{dk}{\gamma}, \tag{3}$$

where $\gamma = (q^2 - k^2)^{1/2}$, $Im(\gamma) \geqq 0$ and $z \geqq 0^+$. This equation represents a cylindrical function as a superposition of plane waves in terms of the horizontal wave number. It is possible, through direct derivations of equation (3), to show that

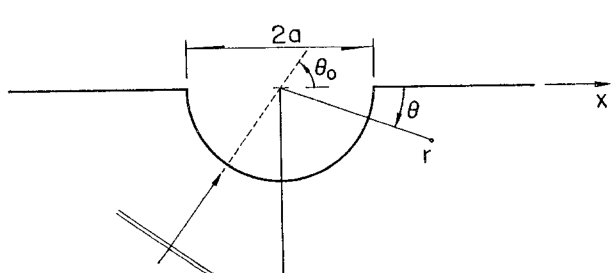

FIG. 1. Geometry of the problem.

$$H_m{}^{(1)}(qr)\cos m\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\left(\frac{\gamma - ik}{q}\right)^m + \left(\frac{-\gamma - ik}{q}\right)^m\right\}\exp(ikx + i\gamma z)\,\frac{dk}{\gamma}. \tag{4}$$

It is also possible to obtain this expression from an integral representation of Hankel function (see equation (8), of Watson, 1958). A similar representation has been used by Scheidl and Ziegler (1978).

A very interesting aspect arises from this integral representation. It is easy to show that the left hand side of equation (4) satisfies the reduced wave equation and the free-boundary condition at $\theta = 0$, π, except at $r = 0$ because of the singularities of Hankel function. That the right-hand side is also a solution is clear, but the fulfillment of zero normal derivative of the integral at $z = 0$ is not evident. It is convenient to note that the expression inside brackets in the integrand is an *integer* polynomial of order m in the horizontal wavenumber k. Therefore, the boundary condition at $z = 0$ is satisfied in the *distribution* sense because

$$\frac{i^n}{2\pi}\int_{-\infty}^{\infty} k^n \exp(ikx)\,dk = \frac{d^n\delta(x)}{dx^n} \tag{5}$$

where $\delta(\cdot)$ = Dirac delta function and n is an integer.

The Exact Solution as an Integral in the Horizontal Wavenumber

From equations (2) and (4) it is possible to write the exact solution as

$$v = \exp(ik_0x - i\gamma_0 z) + \int_{-\infty}^{\infty} A(k)\exp(ikx + i\gamma z)\,dk, \tag{6}$$

where $z \geqq 0^+$, and

$$A(k) = \delta(k - k_0) - \frac{1}{\pi}\sum_{m=0}^{\infty} b_m\left\{\left(\frac{\gamma - ik}{q}\right)^m + \left(\frac{-\gamma - ik}{q}\right)^m\right\}\gamma^{-1}. \tag{7}$$

This expression shows that the diffracted part of the solution can be written in terms of explicitly downgoing waves plus inhomogeneous waves. This makes clear that the upgoing diffracted field admits a representation in terms of only inhomogeneous waves. The complete evaluation of this part of the solution requires dealing with infinite integrals or, if the periodicity is invoked, with infinite sums. It seems convenient at this point to mention that, for a related problem, Millar (1973) established the completeness of the set of plane waves and showed that there is "a linear combination of elements of the set that converges on the boundary to the prescribed values, in the mean square sense as $N \rightarrow \infty$." Furthermore, he established that at points not on the surface, "the expansion converges uniformly to the sought solution whether or not the Rayleigh hypothesis is satisfied."

Conclusions

The difficulty of the Aki and Larner (1970) discrete wavenumber method to accurately represent wave fields close to large slope interfaces is due to the very slow convergence associated to this representation. The Rayleigh ansatz is quite good. We illustrated this fact with one of the few problems that has an exact solution and admits a relatively simple integral representation. In this note, we have shown that this integral contains everything and that the locally upgoing energy is represented by inhomogeneous waves.

The methods of Bouchon (1985), Campillo and Bouchon (1985), and Kawase (1988) are pointed in the appropriate direction as they explicitly include upgoing waves and, in practice, solve the problem. However, in the search for more efficient techniques, it seemed appropriate to point out a rather subtle problem and contribute to a better understanding of a powerful technique.

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